Intro. to ODEs

Quiz 11 **Solutions**

Instructions: Find the general solution to the following systems of linear differential equations. Give your answer in real form!

1)

$$\frac{dx}{dt} = -x - 8y$$
$$\frac{dy}{dt} = 2x - y$$

The eigenvalues are $\lambda = -1 \pm 4i$.

$$\begin{vmatrix} -1-\lambda & -8 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda + 17 = (\lambda+1)^2 + 16$$

For $\lambda = -1 + 4i$, an associated eigenvector is found as follows.

$$\left[\begin{array}{cc|c} -1-(-1\pm 4i) & -8 & 0 \\ 2 & -1-(-1\pm 4i) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \vec{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

We write the solution in real form by pulling apart the complex form into real and imaginary parts.

$$\begin{aligned} e^{(-1+4i)t}\begin{bmatrix} 2i \\ 1 \end{bmatrix} &= e^{-t}(\cos(4t) + i\sin(4t))\begin{bmatrix} 2i \\ 1 \end{bmatrix} \\ &= e^{-t}\begin{bmatrix} -2\sin(4t) \\ \cos(4t) \end{bmatrix} + ie^{-t}\begin{bmatrix} 2\cos(4t) \\ \sin(4t) \end{bmatrix} \end{aligned}$$

This shows us that the general solution is

$$\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -2\sin(4t) \\ \cos(4t) \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2\cos(4t) \\ \sin(4t) \end{bmatrix}.$$

Separating this into components gives us the equivalent form below.

$$x(t) = -2c_1 e^{-t}\sin(4t) + 2c_2 e^{-t}\cos(4t)$$
$$y(t) = c_1 e^{-t}\cos(4t) + c_2 e^{-t}\sin(4t)$$

TURN OVER!

2)

$$\begin{aligned}\frac{dx}{dt} &= 2x + y\\ \frac{dy}{dt} &= 4x + y - 4z\\ \frac{dz}{dt} &= -x + y + 3z\end{aligned}$$

The eigenvalues for this matrix are $\lambda = 1, 2, 3$.

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 4 & 1-\lambda & -4 \\ -1 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2-4\lambda+7)-4(2-\lambda) = (\lambda-1)(2-\lambda)(\lambda-3)$$

For $\lambda = 1$, we find

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 4 & 0 & -4 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda = 2$, we find

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 4 & -1 & -4 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 3$, we find

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 4 & -2 & -4 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \vec{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

This gives us the general solution

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

The equivalent solution in component form is

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{2t} + 2c_3 e^{3t} \\ y(t) &= -c_1 e^t \qquad\quad\; + 2c_3 e^{3t} \\ z(t) &= c_1 e^t + c_2 e^{2t} + c_3 e^{3t}. \end{aligned}$$